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Stability criteria for linear periodic Hamiltonian systems ☆

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ABSTRACT

In this paper, we obtain new stability criteria for linear periodic Hamiltonian systems. A Lyapunov type inequality is established. Our results improve the existing works in the literature.

1. Introduction

Consider the first order linear differential system

$$x' = a(t)x + b(t)u, \qquad u' = -c(t)x - a(t)u, \quad t \in \mathbb{R}, \tag{1.1}$$

where $a(t)$, $b(t)$ and $c(t)$ are real-valued piece-wise continuous functions defined on $\mathbb{R}$. Let

$$y_1(t) = x(t), \qquad y_2(t) = u(t), \qquad y(t) = \big(y_1(t), y_2(t)\big)^\top$$

and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad H(t) = \begin{pmatrix} c(t) & a(t) \\ a(t) & b(t) \end{pmatrix}.$$

Then system (1.1) can be written as a Hamiltonian system

$$y' = JH(t)y, \quad t \in \mathbb{R}. \tag{1.2}$$

We remark that the second order differential equation

$$\big(p(t)x'\big)' + q(t)x = 0, \quad t \in \mathbb{R}, \tag{1.3}$$

can be also written as a system of type (1.1), where $p(t)$ and $q(t)$ are real-valued piece-wise continuous functions and $p(t) \neq 0$ for all $t \in \mathbb{R}$. Indeed, let $x(t)$ be a solution of (1.3) and set $u(t) = p(t)x'(t)$. Then we have

$$x'(t) = \frac{1}{p(t)}u(t), \qquad u'(t) = -q(t)x(t), \quad t \in \mathbb{R}. \tag{1.4}$$

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So, (1.3) is equivalent to (1.1) with

$$a(t) \equiv 0, \qquad b(t) = \frac{1}{p(t)}, \qquad c(t) = q(t).$$

Next, assume that system (1.1) is T-periodic, i.e., the coefficients $a(t)$, $b(t)$ and $c(t)$ satisfy the periodicity conditions

$$a(t+T) = a(t), \qquad b(t+T) = b(t), \qquad c(t+T) = c(t), \quad t \in \mathbb{R}. \tag{1.5}$$

Definition 1.1. The system (1.1) is said to be stable if all solutions are bounded on $\mathbb{R}$, unstable if all nontrivial solutions are unbounded on $\mathbb{R}$, and conditionally stable if there exists a nontrivial solution bounded on $\mathbb{R}$.

In [5], Krein proved the following result (see Sections 7 and 8 therein).

Theorem 1.1. *(See [5].) Assume that*

$$b(t) \geqslant 0, \qquad c(t) \geqslant 0, \qquad b(t)c(t) - a^2(t) \geqslant 0, \quad t \in [0, T]; \tag{1.6}$$

$$\int_0^T b(t)\,dt \int_0^T c(t)\,dt - \left(\int_0^T a(t)\,dt\right)^2 > 0; \tag{1.7}$$

and

$$\int_0^T |a(t)|\,dt + \left(\int_0^T b(t)\,dt\right)^{1/2} \left(\int_0^T c(t)\,dt\right)^{1/2} < 2. \tag{1.8}$$

Then system (1.1) *is stable.*

In [3], the conditions (1.6) and (1.7) of Theorem 1.1 were slightly changed and the following result was proved by a different method.

Theorem 1.2. *(See [3].) Assume that* (1.8) *holds, and that*

$$b(t) > 0, \qquad c(t) \geqslant 0, \qquad b(t)c(t) - a^2(t) \geqslant 0, \quad t \in [0, T]; \tag{1.9}$$

and

$$b(t)c(t) - a^2(t) \not\equiv 0, \quad t \in [0, T]. \tag{1.10}$$

Then system (1.1) *is stable.*

In [4], Guseinov and Zafer established the following theorem which further improves Theorem 1.2, see Corollary 3.1 in [4].

Theorem 1.3. *(See [4].) Assume that*

$$b(t) > 0, \quad t \in [0, T], \qquad \int_0^T \left(c(t) - \frac{a^2(t)}{b(t)}\right) dt > 0; \tag{1.11}$$

and

$$\int_0^T |a(t)|\,dt + \left(\int_0^T b(t)\,dt\right)^{1/2} \left(\int_0^T c^+(t)\,dt\right)^{1/2} < 2. \tag{1.12}$$

Then system (1.1) *is stable, where and in the sequel* $c^+(t) = \max\{c(t), 0\}$.

Remark 1.1. In fact, they used the following condition

$$\int_0^T |a(t)|\,dt + \left(\int_0^T b(t)\,dt\right)^{1/2} \left(\int_0^T c^+(t)\,dt\right)^{1/2} \leqslant 2 \tag{1.13}$$

instead of (1.12) in Corollary 3.1 of [4]. However, in view of the proof (from line 4 to line 5 on page 1204) of Lemma 4.3 in [4], the above condition (1.13) should be replaced by (1.12). Only when $a(t) \equiv 0$, i.e. in case the Hamiltonian system (1.1) reduces to the second order differential equation (1.3), the weaker inequality (1.13) will suffice.

In the present paper, we will establish new stability criteria for the periodic system (1.1). As a simple corollary of our main results, we obtain a new condition different from (1.12)

$$\int_0^T b(t)\,dt \int_0^T c^+(t)\,dt < 4\exp\left(-2\int_0^T \left|a(s)\right| ds\right). \tag{1.14}$$

It is easy to see that condition (1.12) fails while condition (1.14) still takes effect when $\int_0^T |a(t)|\,dt \geqslant 2$. In addition, if there exists a constant $\gamma > 0$ such that $b(t) \leqslant \gamma |a(t)|$ for $t \in [0, T]$, then we can deduce another new condition different from (1.12)

$$\gamma \int_0^T c^+(t)\,dt < \frac{8}{\exp(2\int_0^T |a(s)|\,ds) - 1}, \tag{1.15}$$

which improves condition (1.12), see Remark 4.2 is Section 4.

The paper is organized as follows. In Section 2, we outline some basic facts about Floquet theory for system (1.1). In Section 3, we derive a Lyapunov inequality for Hamiltonian system (1.1) which is closely related to the stability criteria obtained in this paper. The main results of the paper, stability criteria, are stated and proved in Section 4.

2. Preliminaries

Let

$$X(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ u_1(t) & u_2(t) \end{pmatrix}, \qquad X(0) = I_2$$

be a fundamental matrix solution of (1.1). Under the periodicity conditions (1.5), the Floquet theory holds. For details of the Floquet theory we refer to [2,6]. The Floquet multipliers (real or complex) of (1.1) are the roots of

$$\det\bigl(\lambda I_2 - X(T)\bigr) = 0,$$

which is equivalent to

$$\lambda^2 - A\lambda + 1 = 0, \tag{2.1}$$

where

$$A = x_1(T) + u_2(T). \tag{2.2}$$

It follows from the Floquet theory that corresponding to each (complex) root λ there is a nontrivial solution $y(t) = (x(t), u(t))^\top$ of (1.1) such that

$$y(t+T) = \lambda y(t), \quad t \in \mathbb{R}. \tag{2.3}$$

Note that if λ_1 and λ_2 are the Floquet multipliers, then we have

$$\lambda_1 + \lambda_2 = A, \qquad \lambda_1\lambda_2 = 1.$$

System (1.1) has two linearly independent solutions and any solution of (1.1) can be written as their linear combination. In particular, if $\lambda_1 \neq \lambda_2$, then system (1.1) has two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$ such that

$$\phi_1(t+T) = \lambda_1\phi_1(t), \qquad \phi_2(t+T) = \lambda_2\phi_2(t), \quad t \in \mathbb{R}. \tag{2.4}$$

Lemma 2.1. *(See [4].) System* (1.1) *is unstable if* $|A| > 2$*, and stable if* $|A| < 2$*.*

Lemma 2.2. *(See [4].) If* $|A| = 2$*, then system* (1.1) *is stable when* $u_1(T) = x_2(T) = 0$*; but conditionally stable and not stable otherwise.*

3. Lyapunov inequality

In this section we establish a new Lyapunov type inequality related to system (1.1). Lyapunov inequalities are useful in oscillation, disconjugacy, and boundary value problems, see [1]. As far as a Lyapunov type inequality is concerned, system (1.1) is not required to be periodic.

Theorem 3.1. *Let $b(t) \geqslant 0$ for $t \in \mathbb{R}$. Assume that* (1.1) *has a real solution $(x(t), u(t))$ such that $x(t_1) = x(t_2) = 0$ and $x(t)$ is not identically zero on $[t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Then the Lyapunov inequality*

$$\left[\int_{t_1}^{t_2} b(t) \exp\left(-2\int_{\tau}^{t} a(s)\,ds\right) dt\right] \int_{t_1}^{t_2} c^+(t)\,dt \geqslant 4 \tag{3.1}$$

holds for some $\tau \in (t_1, t_2)$.

Proof. Multiplying the first equation of (1.1) by $u(t)$ and the second one by $x(t)$, and then adding the results, we obtain

$$\big(x(t)u(t)\big)' = b(t)u^2(t) - c(t)x^2(t).$$

Integrating the above equation from t_1 to t_2 and taking into account that $x(t_1) = x(t_2) = 0$ yields

$$\int_{t_1}^{t_2} b(t)u^2(t)\,dt = \int_{t_1}^{t_2} c(t)x^2(t)\,dt. \tag{3.2}$$

Choose $\tau \in (t_1, t_2)$ such that

$$\big|x(\tau)\big| = \max_{t_1 \leqslant t \leqslant t_2} \big|x(t)\big|.$$

Since $x(t)$ is not identically zero on $[t_1, t_2]$, we have $|x(\tau)| > 0$. From the first equation of (1.1), we have

$$\left[x(t) \exp\left(-\int_{t_1}^{t} a(s)\,ds\right)\right]' = b(t)u(t) \exp\left(-\int_{t_1}^{t} a(s)\,ds\right), \quad t \in \mathbb{R}, \tag{3.3}$$

and

$$\left[x(t) \exp\left(\int_{t}^{t_2} a(s)\,ds\right)\right]' = b(t)u(t) \exp\left(\int_{t}^{t_2} a(s)\,ds\right), \quad t \in \mathbb{R}. \tag{3.4}$$

Integrating (3.3) from t_1 to τ and taking into account that $x(t_1) = 0$, we get

$$x(\tau) = \int_{t_1}^{\tau} b(t)u(t) \exp\left(-\int_{\tau}^{t} a(s)\,ds\right) dt.$$

It follows that

$$\big|x(\tau)\big| \leqslant \int_{t_1}^{\tau} b(t)\big|u(t)\big| \exp\left(-\int_{\tau}^{t} a(s)\,ds\right) dt. \tag{3.5}$$

Similarly, integrating (3.4) from τ to t_2 and taking into account that $x(t_2) = 0$, we get

$$\big|x(\tau)\big| \leqslant \int_{\tau}^{t_2} b(t)\big|u(t)\big| \exp\left(-\int_{\tau}^{t} a(s)\,ds\right) dt. \tag{3.6}$$

Adding (3.5) and (3.6), applying the Cauchy–Schwarz inequality and using (3.2), we have

$$2\big|x(\tau)\big| \leqslant \int_{t_1}^{t_2} b(t)\big|u(t)\big| \exp\left(-\int_{\tau}^{t} a(s)\,ds\right) dt$$

$$\leqslant \left[\int_{t_1}^{t_2} b(t)\exp\left(-2\int_{\tau}^{t} a(s)\,ds\right)dt\right]^{1/2}\left[\int_{t_1}^{t_2} b(t)u^2(t)\,dt\right]^{1/2}$$
$$= \left[\int_{t_1}^{t_2} b(t)\exp\left(-2\int_{\tau}^{t} a(s)\,ds\right)dt\right]^{1/2}\left[\int_{t_1}^{t_2} c(t)x^2(t)\,dt\right]^{1/2}$$
$$\leqslant \left|x(\tau)\right|\left[\int_{t_1}^{t_2} b(t)\exp\left(-2\int_{\tau}^{t} a(s)\,ds\right)dt\right]^{1/2}\left[\int_{t_1}^{t_2} c^+(t)\,dt\right]^{1/2}.$$

It follows that

$$\left[\int_{t_1}^{t_2} b(t)\exp\left(-2\int_{\tau}^{t} a(s)\,ds\right)dt\right]^{1/2}\left[\int_{t_1}^{t_2} c^+(t)\,dt\right]^{1/2} \geqslant 2.$$

The proof is complete. □

4. Stability criteria

Lemma 4.1. *(See [4].) Assume that*

$$b(t) > 0, \quad t \in [0, T], \qquad \int_0^T \left(c(t) - \frac{a^2(t)}{b(t)}\right)dt > 0. \tag{4.1}$$

If $A^2 \geqslant 4$, then system (1.1) has a nontrivial solution $(x(t), u(t))$ possessing the following properties: there exist $t_1, t_2 \in \mathbb{R}$ such that

$$0 \leqslant t_1 \leqslant T, \qquad t_1 < t_2 \leqslant T + t_1, \qquad x(t_1) = x(t_2) = 0, \qquad x(t) \neq 0 \quad \text{for all } t \in (t_1, t_2). \tag{4.2}$$

Lemma 4.2. *(See [4].) Assume that*

$$b(t) > 0, \quad t \in [0, T], \qquad \int_0^T \left(c(t) - \frac{a^2(t)}{b(t)}\right)dt = 0; \tag{4.3}$$

and

$$a/b \notin C^1(0, T), \quad \text{or} \quad a/b \in C^1(0, T), \qquad \left(\frac{a(t)}{b(t)}\right)' - c(t) + \frac{a^2(t)}{b(t)} \not\equiv 0, \quad t \in [0, T]. \tag{4.4}$$

If $A^2 \geqslant 4$, then the conclusion of Lemma 4.1 remains valid.

Theorem 4.1. *Assume that (4.1) holds, and that*

$$\left[\max_{\tau,\sigma \in [0,T]} \int_0^T b(t+\sigma)\exp\left(-2\int_{\tau}^{t} a(s+\sigma)\,ds\right)dt\right]\int_0^T c^+(t)\,dt < 4. \tag{4.5}$$

Then system (1.1) is stable.

Proof. In view of Lemma 2.1, it is sufficient to show that $A^2 < 4$. Assume on the contrary that $A^2 \geqslant 4$. Then by Lemma 4.1, system (1.1) has a nontrivial solution $(x(t), u(t))$ such that (4.2) holds. Applying Theorem 3.1, we see that inequality (3.1) holds for some $\tau \in (t_1, t_2)$. It follows from (1.5) and (4.2) that

$$4 \leqslant \left[\int_{t_1}^{t_2} b(t)\exp\left(-2\int_{\tau}^{t} a(s)\,ds\right)dt\right]\int_{t_1}^{t_2} c^+(t)\,dt$$
$$\leqslant \left[\int_{t_1}^{t_1+T} b(t)\exp\left(-2\int_{\tau}^{t} a(s)\,ds\right)dt\right]\int_{t_1}^{t_1+T} c^+(t)\,dt$$

$$= \left[\int_0^T b(t+t_1) \exp\left(-2 \int_{\tau - t_1}^{t} a(s+t_1)\, ds \right) dt \right] \int_0^T c^+(t)\, dt$$

$$\leqslant \left[\max_{\tau,\sigma \in [0,T]} \int_0^T b(t+\sigma) \exp\left(-2 \int_{\tau}^{t} a(s+\sigma)\, ds \right) dt \right] \int_0^T c^+(t)\, dt,$$

which contradicts condition (4.5). Thus $A^2 < 4$ and hence system (1.1) is stable. This completes the proof. □

Theorem 4.2. *Assume that* (1.5), (4.3), (4.4) *and* (4.5) *hold. Then system* (1.1) *is stable.*

The proof of Theorem 4.2 is exactly the same as that of Theorem 4.1, so we omit it.
For any $\sigma, \tau \in [0, T]$, we have

$$\int_0^T b(t+\sigma) \exp\left(-2 \int_{\tau}^{t} a(s+\sigma)\, ds \right) dt$$

$$\leqslant \int_0^{\tau} b(t+\sigma) \exp\left(2 \int_t^{\tau} \left| a(s+\sigma) \right| ds \right) dt + \int_{\tau}^{T} b(t+\sigma) \exp\left(2 \int_{\tau}^{t} \left| a(s+\sigma) \right| ds \right) dt$$

$$\leqslant \exp\left(2 \int_0^{\tau} \left| a(s+\sigma) \right| ds \right) \int_0^{\tau} b(t+\sigma)\, dt + \exp\left(2 \int_{\tau}^{T} \left| a(s+\sigma) \right| ds \right) \int_{\tau}^{T} b(t+\sigma)\, dt$$

$$\leqslant \exp\left(2 \int_0^{T} \left| a(s+\sigma) \right| ds \right) \int_0^{T} b(t+\sigma)\, dt$$

$$= \exp\left(2 \int_0^{T} \left| a(s) \right| ds \right) \int_0^{T} b(t)\, dt.$$

This shows that

$$\max_{\tau,\sigma \in [0,T]} \int_0^T b(t+\sigma) \exp\left(-2 \int_{\tau}^{t} a(s+\sigma)\, ds \right) dt \leqslant \exp\left(2 \int_0^{T} \left| a(s) \right| ds \right) \int_0^{T} b(t)\, dt.$$

Thus, in view of Theorems 4.1 and 4.2, we have the following corollaries.

Corollary 4.1. *Assume that* (4.1) *holds, and that*

$$\int_0^T b(t)\, dt \int_0^T c^+(t)\, dt < 4 \exp\left(-2 \int_0^T \left| a(s) \right| ds \right). \tag{4.6}$$

Then system (1.1) *is stable.*

Corollary 4.2. *Assume that* (4.3), (4.4) *and* (4.6) *hold. Then system* (1.1) *is stable.*

Remark 4.1. When $\int_0^T |a(t)|\, dt < 2$, we can rewrite (1.8) as

$$\int_0^T b(t)\, dt \int_0^T c^+(t)\, dt < \left(2 - \int_0^T \left| a(t) \right| dt \right)^2. \tag{4.7}$$

It is easy to see that condition (4.6) is stronger than (4.7) when $\int_0^T |a(t)|\, dt < \delta_0 \approx 1.594\ldots$, but condition (4.6) is weaker than (4.7) when $\int_0^T |a(t)|\, dt > \delta_0 \approx 1.594\ldots$, where δ_0 is the unique positive root of the equation $(2-x)e^x = 2$. In particular, condition (4.7) (or (1.8)) fails while condition (4.6) still takes effect when $\int_0^T |a(t)|\, dt \geqslant 2$.

If there exists a constant $\gamma > 0$ such that $b(t) \leqslant \gamma |a(t)|$ for $t \in [0, T]$, then for any $\sigma, \tau \in [0, T]$, we have

$$\begin{aligned}
&\int_0^T b(t+\sigma)\exp\left(-2\int_\tau^t a(s+\sigma)\,ds\right)dt \\
&\quad\leqslant \gamma \int_0^T \left|a(t+\sigma)\right| \exp\left(-2\int_\tau^t a(s+\sigma)\,ds\right)dt \\
&\quad\leqslant \gamma \left[\int_0^\tau \left|a(t+\sigma)\right| \exp\left(2\int_t^\tau \left|a(s+\sigma)\right| ds\right)dt + \int_\tau^T \left|a(t+\sigma)\right| \exp\left(2\int_\tau^t \left|a(s+\sigma)\right| ds\right)dt\right] \\
&\quad= \frac{\gamma}{2}\left[\exp\left(2\int_0^\tau \left|a(s+\sigma)\right| ds\right) + \exp\left(2\int_\tau^T \left|a(s+\sigma)\right| ds\right) - 2\right] \\
&\quad= \frac{\gamma}{2}\left\{-\left[\exp\left(2\int_0^\tau \left|a(s+\sigma)\right| ds\right) - 1\right]\left[\exp\left(2\int_\tau^T \left|a(s+\sigma)\right| ds\right) - 1\right] + \exp\left(2\int_0^T \left|a(s+\sigma)\right| ds\right) - 1\right\} \\
&\quad\leqslant \frac{\gamma}{2}\left[\exp\left(2\int_0^T \left|a(s+\sigma)\right| ds\right) - 1\right] \\
&\quad= \frac{\gamma}{2}\left[\exp\left(2\int_0^T \left|a(s)\right| ds\right) - 1\right],
\end{aligned}$$

which yields that

$$\max_{\tau,\sigma\in[0,T]} \int_0^T b(t+\sigma)\exp\left(-2\int_\tau^t a(s+\sigma)\,ds\right)dt \leqslant \frac{\gamma}{2}\left[\exp\left(2\int_0^T \left|a(s)\right| ds\right) - 1\right].$$

Thus, in view of Theorems 4.1 and 4.2, we have the following corollaries.

Corollary 4.3. *Assume that*

$$\gamma\left|a(t)\right| \geqslant b(t) > 0, \quad t \in [0, T], \qquad \int_0^T \left(c(t) - \frac{a^2(t)}{b(t)}\right)dt > 0; \tag{4.8}$$

and

$$\gamma \int_0^T c^+(t)\,dt < \frac{8}{\exp(2\int_0^T |a(s)|\,ds) - 1}. \tag{4.9}$$

Then system (1.1) *is stable.*

Corollary 4.4. *Assume that* (4.4) *and* (4.9) *hold, and that*

$$\gamma\left|a(t)\right| \geqslant b(t) > 0, \quad t \in [0, T], \qquad \int_0^T \left(c(t) - \frac{a^2(t)}{b(t)}\right)dt = 0. \tag{4.10}$$

Then system (1.1) *is stable.*

Set $g(x) = 1 + x - (1-x)e^{2x}$ for $x \in [0, 1]$. Then

$$g'(x) = 1 + (2x-1)e^{2x}, \qquad g''(x) = 4e^{2x} > 0, \quad x \in [0, 1].$$

It follows that $g'(x) > g'(0) = 0$ for $x \in (0, 1]$, and so

$$g(x) = 1 + x - (1 - x)e^{2x} > g(0) = 0, \quad x \in (0, 1].$$

This shows that

$$e^{2x} < \frac{1 + x}{1 - x}, \quad x \in (0, 1).$$

Set $y = 2x$, then it follows that

$$e^{2y} - 1 < \frac{8y}{(2 - y)^2}, \quad 0 < y < 2. \tag{4.11}$$

Remark 4.2. When $\int_0^T |a(t)|\,dt < 2$ and $b(t) \leqslant \gamma |a(t)|$ for $t \in [0, T]$, we can rewrite (1.8) as

$$\gamma \int_0^T c^+(t)\,dt < \frac{(2 - \int_0^T |a(t)|\,dt)^2}{\int_0^T |a(t)|\,dt}. \tag{4.12}$$

By (4.11), we have

$$\frac{(2 - \int_0^T |a(t)|\,dt)^2}{\int_0^T |a(t)|\,dt} < \frac{8}{\exp(2\int_0^T |a(s)|\,ds) - 1} \quad \text{for } 0 < \int_0^T \big|a(t)\big|\,dt < 2.$$

Therefore, condition (4.9) is weaker than (4.12). In particular, condition (4.12) (or (1.8)) fails while condition (4.9) still takes effect when $\int_0^T |a(t)|\,dt \geqslant 2$.

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